A Large N Dual

For 4d QCD

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A Large N Dual

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A Large N Dual

For 2d CFTs

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Based on

- Matthias Gaberdiel, Rajesh Gopakumar and Arunabha Saha (arXiv:1009.6087).
- Matthias Gaberdiel and Rajesh Gopakumar (arXiv:1011.2986).
- M. R. Gaberdiel, R. Gopakumar, T. Hartman and S. Raju (work in progress).

Outline of the talk

- Motivation and Introductory Remarks
  - Statement of the Duality
- The Minimal Model CFTs
  - $SU(N)$ Coset WZW theories and their large N limit
- Higher Spin Symmetries
  - In the Bulk and in the Boundary
- Further Checks of the Proposal
  - Matching of Partition Functions and RG Flow Pattern
- Towards a Derivation
  - Simplest Realisation of Holography?
- Where To?

1 Motivation and Introductory Remarks

Motivation

- Common Complaint: AdS/CFT works wonderfully in all these SUSY theories.
- But what about regular CFTs/QFTs?
- How do we know that holography is not something special to SUSY theories?.
- To what extent do we need to have an embedding in a complete string theory?
- 2d CFTs are archetypes of well understood non-trivial QFTs. Can we study a large N family and find a gravity dual?
- **Yes**. For a class of $SU(N)$ WZW coset models (which includes the usual Virasoro minimal models as a special case).
- Exploit a *different* symmetry - higher spin algebra. (Fradkin-Vasiliev).
- In many ways analogous to the Klebanov-Polyakov conjecture for the $O(N)$ model - but with additional 'tHooft coupling λ.

The Duality

- The CFT: a coset WZW model.

$$\frac{SU(N)_k \times SU(N)_1}{SU(N)_{k+1}}$$

- Take the 't Hooft large N limit, keeping $0 \leq \lambda = \frac{N}{N+k} \leq 1$ fixed.
- A family of theories with central charge $c_N(\lambda) = N(1-\lambda^2)$ - vector like model.
- The Bulk Dual: A Vasiliev type higher spin theory (including spins $2, 3 \ldots \infty$) coupled to two equally massive complex scalar fields.
- $M^2 = -(1-\lambda^2)$. But the two scalars have opposite quantisation.
- λ also plays the role of a deformation parameter of the higher spin algebra ("like α'"). Also $G_N \propto \frac{1}{N}$.

2 The Minimal Model CFTs

Coset Models

- A G/H coset theory is defined in terms of a G WZW theory in which a subgroup H is gauged (without kinetic term).

- Therefore $T_{G/H}(z) = T_G(z) - T_H(z)$ and $c_{G/H} = c_G - c_H$.

- Building block for rational CFTs for different G and H.

- Basic case: $G = SU(N)_k \times SU(N)_l$ and $H = SU(N)_{k+l}$ (diagonal).

- We will consider the case $l = 1$ (in the large N limit, additional l is like adding flavour).

- So-called $\mathcal{W}_N$ minimal model series (since it possesses a $\mathcal{W}_N$ symmetry).

Coset Models *continued*

✧ Thus the class of models to focus on is $\frac{SU(N)_k \times SU(N)_1}{SU(N)_{k+1}}$.

✧ For this family,

$$c_N(k) = (N-1)[1 - \frac{N(N+1)}{p(p+1)}] \leq (N-1)$$

where $p = k + N$. i.e. $(p = N+1, N = 2, \ldots)$

✧ In the case $N = 2$, this is the coset construction of the unitary Virasoro discrete series (GKO). $c_2(k) = 1 - \frac{6}{p(p+1)} \leq 1$ with $p = 3, 4 \ldots$.

✧ Special cases - $k = 1$: Z_N parafermion theory describes Z_N ising model at criticality. $c_N(1) = \frac{2(N-1)}{N+2}$.

✧ Special cases - $k = \infty$: $c_N(\infty) = (N-1)$. Delicate limit to take. Theory of $(N-1)$ free bosons.

Coset Models *Spectrum*

- Spectrum of Primaries are labelled by two integrable representations (Λ^+, Λ^-) of $SU(N)_k$ and $SU(N)_{k+1}$ respectively.

- Dimensions given by:

$$h(\Lambda^+;\Lambda^-) = \frac{1}{2p(p+1)}\left(\Big|(p+1)(\Lambda^+ + \rho) - p(\Lambda^- + \rho)\Big|^2 - \rho^2\right)$$

  ρ is the Weyl vector for $SU(N)$.

- In the case, $N = 2$ reduces to usual Friedan-Shenker-Qiu expression

$$h(r,s) = \frac{(r(p+1) - sp)^2 - 1}{4p(p+1)} = h(p-r, p+1-s)$$

- Particular cases: $h(0;\mathrm{f}) = \frac{(N-1)}{2N}\left(1 - \frac{N+1}{N+k+1}\right)$; $\quad h(\mathrm{f};0) = \frac{(N-1)}{2N}\left(1 + \frac{N+1}{N+k}\right)$

- $h(0;\mathrm{adj}) = 1 - \frac{N}{N+k+1}$; $\quad h(\mathrm{adj};0) = 1 + \frac{N}{N+k}$

Coset Models *Characters*

- The partition function of the coset theory given in terms of contributions from each of these primaries.

- Captured by "branching functions" in the decomposition of G WZW characters in terms of those for H.

-
$$\chi_{(\Lambda^+;\Lambda^-)}(q) = \frac{1}{\eta(q)^{N-1}} \sum_{w\in\hat{W}} \epsilon(w) q^{\frac{1}{2p(p+1)}((p+1)w(\Lambda^+ + \rho) - p(\Lambda^- + \rho))^2}.$$

  $\hat{W}$ is the affine Weyl group (affine translations +usual Weyl reflections).

- Analogue of Rocha-Caridi characters for Virasoro minimal models.

- (Diagonal) modular invariant partition function given by
$$Z_{CFT} = \sum_{\Lambda^+,\Lambda^-} |\chi_{(\Lambda^+;\Lambda^-)}(q)|^2.$$

Coset Models *RG flows*

- One can flow between the minimal models with different k or p (for fixed N).
- The most relevant operator of the pth minimal model, $(0; \mathrm{adj})$, induces the RG flow. The IR fixed point is the $p-1$th model.
- $$(0; \mathrm{adj})_p \qquad \xrightarrow{\text{RG-flow by } (0;\, \mathrm{adj})} \qquad (\mathrm{adj}; 0)_{p-1}.$$
- Analogue of $(1, 3)$ operator flowing to $(3, 1)$ operator for Virasoro minimal models.
- Similar analogues of $(1, 2)$ operator flowing to $(2, 1)$ operators

$$(0; \mathrm{f})_p \qquad \xrightarrow{\text{RG-flow by } (0;\, \mathrm{adj})} \qquad (\mathrm{f}; 0)_{p-1}$$

$$(0; \bar{\mathrm{f}})_p \qquad \xrightarrow{\text{RG-flow by } (0;\, \mathrm{adj})} \qquad (\bar{\mathrm{f}}; 0)_{p-1}.$$

Coset Models *'tHooft limit*

- Now take the 'tHooft limit $N, k \to \infty$ with $\lambda = \frac{N}{k+N}$ fixed.
- The central charge $c_N(\lambda) \simeq N(1-\lambda^2) < N$
- Dimensions of operators simplify remarkably:

$$h(0;\mathrm{f}) = \frac{1}{2}(1-\lambda)\ , \qquad h(\mathrm{f};0) = \frac{1}{2}(1+\lambda).$$

-

$$h(0;\mathrm{adj}) = 1-\lambda\ , \qquad h(\mathrm{adj};0) = 1+\lambda.$$

- In general, representations which are finite tensor products of fund./antifund. have finite dimension.
- But there are several primaries that have degenerate dimensions e.g. (R, R) operators

$$h_{(R,R)} = \frac{B(R)}{2} \times \frac{\lambda^2}{N},$$

($B(R)$ is the number of boxes in the young tableaux corresponding to R.)

3 Higher Spin Symmetries

Higher Spin Symmetries *boundary*

- The $SU(N)$ cosets have an extended $\mathcal{W}_N$ symmetry.

- In addition to $T(z)$, higher spin conserved currents $W^{(3)}(z), \ldots W^{(N)}(z)$.

- Constructed using higher order Casimir invariants.

- E.g $W^{(3)}(z) \propto d^{abc}[a_1(J^a_{(1)}J^b_{(1)}J^c_{(1)})(z) + a_2(J^a_{(2)}J^b_{(1)}J^c_{(1)})(z) + a_3(J^a_{(2)}J^b_{(2)}J^c_{(1)})(z) + a_4(J^a_{(2)}J^b_{(2)}J^a_{(2)})(z)]$.

- Similarly, $W^{(m)}(z)$ from mth order Casimir combinations of the currents $J^a_{(1,2)}(z)$ of $SU(N)_k$ and $SU(N)_1$ respectively.

- The $\mathcal{W}_N$ OPE gives rise to a non-linear symmetry algebra - rather than a Lie Algebra.

Higher Spin Symmetries *bulk*

- Unlike flat space, AdS admits consistent classical theories of interacting (massless) higher spin *gauge* fields. (Fradkin-Vasiliev)

- Typically, need an *infinite* tower of these fields. The classical action is also non-local and involves an infinite number of derivatives (See Xi's talk).

- Exception is AdS_3 where one can truncate to a theory with spins $2, 3 \ldots N$. The gauge fields described by an $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ Chern-Simons theory.

- A generalisation of the $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ description of pure AdS_3 gravity. The higher spin fields also have no propagating degrees of freedom.

- Gauge fields: $\varphi_{\mu_1 \mu_2 \cdots \mu_s} \sim \varphi_{\mu_1 \mu_2 \cdots \mu_s} + \nabla_{(\mu_1} \xi_{\mu_2 \cdots \mu_s)}$.

- Express in terms of generalised vielbeins and connections: $e_\mu^{a_1 \cdots a_{s-1}}, \omega_\mu^{a_1 \cdots a_{s-1}}$.

Higher Spin Symmetries *bulk*

- One forms the $SL(N,\mathbb{R})$ gauge fields $A, \tilde{A}$ from combinations of these vielbeins and connections.

- Action is $S = S_{\rm CS}[A] - S_{\rm CS}[\tilde{A}]$ with level $k_{CS} = \frac{\ell}{4G_N}$.

- Brown-Henneaux type analysis of the asymptotic symmetry algebra (Campoleoni et.al., Henneaux-Rey) shows that one gets precisely a boundary $\mathcal{W}_N$ algebra.

- The central charge is *exactly* the same as Brown-Henneaux: $c = \bar{c} = \frac{3\ell}{2G_N}$.

- This analysis is classical. At the quantum level, one can do a one loop calculation for the quadratic fluctuations of the higher spin gauge fields (M.G., R. G. and A.S.).

- For spin-s field obtain: $Z^{(s)} = \left[\det\left(-\Delta + \frac{s(s-3)}{\ell^2}\right)^{\rm TT}_{(s)}\right]^{-\frac{1}{2}} \left[\det\left(-\Delta + \frac{s(s-1)}{\ell^2}\right)^{\rm TT}_{(s-1)}\right]^{\frac{1}{2}}$

Higher Spin Symmetries *bulk*

- Can evaluate to give:

$$Z^{(s)} = \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2}.$$

- Putting together the contributions for spins $s = 2, 3 \ldots \infty$

$$Z_{hs(1.1)} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} = \prod_{n=1}^{\infty} |1-q^n|^2 \times \prod_{n=1}^{\infty} \frac{1}{|(1-q^n)^n|^2} = \prod_{n=1}^{\infty} |1-q^n|^2 \times |M(q)|^2.$$

- This is the vacuum character for $\mathcal{W}_\infty$. ($W_k^{(s)}|0\rangle = 0$ for $k = 0 \ldots s$.)

- Can be argued to be perturbatively exact (adapting Maloney-Witten argument for pure gravity).

- Thus $\mathcal{W}_N$ symmetry can be realised in the bulk at the quantum level.

Higher Spin Symmetries *bulk*

- For the bulk duals to the coset CFTs we have not just the higher spin fields.
- In AdS_3 can also have an *additional* multiplet with scalars/fermions (consistent with the higher spin symmetry).
- Masses depend on a single parameter which also governs the structure of interactions.
- In our case, we need two complex scalars with $M^2 = -(1-\lambda^2)$.
- However, these can be quantised in two alternate ways.
- We need one of them in $(+)$ quantisation (corresponding to $h_+ = \frac{1}{2}(1+\lambda)$) and the other in $(-)$ quantisation (corresponding to $h_- = \frac{1}{2}(1-\lambda)$).
- Thus we will consider a bulk theory with a tower of massless fields together with the two complex scalars above.

4 Checks of the Proposal

Checks of the Proposal

☆ We have already seen the zeroth order check which is the matching of the $\mathcal{W}_N$ symmetries (More general picture of matching $\mathcal{W}_\infty(\lambda)$ symmetries).

☆ Next perform checks on the matching of the spectrum.

☆ Generalised boundary gravitons captured by the MacMahon function:

$$\prod_{s=2}^{\infty}\prod_{n=s}^{\infty}\frac{1}{|1-q^n|^2}=\prod_{n=1}^{\infty}|1-q^n|^2\times\prod_{n=1}^{\infty}\frac{1}{|(1-q^n)^n|^2}.$$

☆ This is accounted for in the CFT by the character of the vacuum representation $|\chi_{(0;0)}(q)|^2$ (the first null vectors are at level $k+1$).

☆ We also have in the bulk fields corresponding to operators of dimension $h_\pm = \frac{1}{2}(1\pm\lambda)$

☆ These correspond to the primaries $(0;f)$ and $(f;0)$ as well as their complex conjugates.

Checks of the Proposal *spectrum*

☆ More generally, in the bulk we have the one loop answer

$$Z^{(1)}_{\rm total} = \prod_{s=2}^{\infty}\prod_{n=s}^{\infty}\frac{1}{|1-q^n|^2} \times \prod_{l_1=0,l_1'=0}^{\infty}\frac{1}{(1-q^{h_-+l_1}\bar{q}^{h_-+l_1'})^2} \times \prod_{l_2=0,l_2'=0}^{\infty}\frac{1}{(1-q^{h_++l_2}\bar{q}^{h_++l_2'})^2}$$

☆ The last two terms come from determinants of each of the two complex scalars.

☆ The claim is that this is equal to the CFT partition function in the $N=\infty$ 'tHooft limit.

$$Z(\lambda) = \lim_{N\to\infty}\sum_{\Lambda^+,\Lambda^-}|\chi_{(\Lambda^+;\Lambda^-)}(q)|^2 .$$

☆ Recall that the characters are given by:

$$\chi_{(\Lambda^+;\Lambda^-)}(q) = \frac{1}{\eta(q)^{N-1}}\sum_{w\in\hat{W}}\epsilon(w)q^{\frac{1}{2p(p+1)}((p+1)w(\Lambda^++\rho)-p(\Lambda^-+\rho))^2} .$$

where $p = k+N$.

Checks of the Proposal *spectrum*

☆ CFT answer simplifies remarkably in the $N = \infty$ 'tHooft limit.

☆ One can show for representations Λ^+, Λ^- which are finite tensor products of fund.anti-fund.(M.G., R.G.,T.H., S.R.)

$$\chi_{(\Lambda^+;\Lambda^-)}(q) \rightarrow q^{C_2(\Lambda^+)} dim_q(\Lambda^+) q^{C_2(\Lambda^-)} dim_q(\Lambda^-) P(\Lambda^-, \Lambda^+) \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2}.$$

☆ $q^{C_2(\Lambda^+)} dim_q(\Lambda^+)$ is essentially a $U(N)$ character in the large N limit.

☆ The sum over Λ^+, Λ^- has the structure of the bulk scalar determinants. (Sum over subsets of representations yields individually the four factors, and various product combinations.)

☆ But need to understand better the contributions of the nearly degenerate representations (e.g. $\chi_{(\Lambda;\Lambda)}(q)$) where the characters become reducible.

Checks of the Proposal *RG flows*

☆ There is an RG flow between the model labelled by p to $(p-1) \Rightarrow \delta\lambda = \frac{\lambda^2}{N}$ and $\delta c = -2\lambda^2$.

☆ The perturbing operator involving $(0; adj)$ (with $h = \bar{h} = 1 - \lambda$) can be written as

$$S_{\rm pert} = g \int d^2z \; \mathcal{O}\,\mathcal{O}^\dagger$$

where $\mathcal{O} = (0; \mathrm{f})$ has $h = \bar{h} = \frac{1}{2}(1-\lambda)$.

☆ Thus a double trace perturbation in the boundary theory.

☆ Bulk interpretation in such cases: the scalar field corresponding to $\mathcal{O}$ is in $(-)$-quantisation in the UV with dimension Δ_-.

☆ Flows to a theory with the $(+)$-quantisation in the IR where it corresponds to an irrelevant operator $\mathcal{O}'$ with dimension $\Delta_+ = 2 - \Delta_-$.

☆ Matches exactly what we see in the boundary theory where $\mathcal{O}' = (\mathrm{f}; 0)$.

5 Towards a Derivation

Towards a Derivation

☆ Can we derive this duality from "first principles"?

☆ Will give a sketch: key is the bulk $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$ Chern-Simons description for the higher spin fields.

☆ No bulk propagating degrees of freedom. Boundary conditions crucial.

☆ The usual boundary conditions are $A_{\tilde{w}} = 0$, $\tilde{A}_w = 0$.

☆ Gauge fields are effectively in $SU(N)$. Might seem to give a $SU(N) \times SU(N)$ WZW theory.

☆ However, additional fall off conditions necessary for asymptotically AdS_3 geometry.

Towards a Derivation *continued*

☆ Essentially, sets upper (lower) triangular components of $A, (\tilde{A})$ to zero.

☆ Amounts to a gauging of the WZW model - Drinfeld-Sokolov reduction.

☆ This was classical. At the quantum level, the DS reduction is more involved (Balog et.al.).

☆ In fact, the quantum version is *exactly* equivalent to our coset CFT with a specific relation between k_{CS} and k.

☆ This gives the link between the higher spin theory and the coset.

☆ But need to understand better the interpretation of the scalar fields in the Chern-Simons picture - Wilson loops?

6 Where to?

Where to?

- ☆ A large family of non-SUSY examples of AdS/CFT with a fairly explicit description in both bulk and boundary.
- ☆ Intermediate complexity between pure gravity and full fledged string theory.
- ☆ Applications to real systems: $\mathbb{Z}_N$ Ising model and other stat. mech. models. (Jimbo, Miwa et,al.)
- ☆ Black holes: Tangible context for microscopics, dynamics.
- ☆ Proving the duality: Nuts and bolts of Holography.
- ☆ 2d QCD: A bulk dual for the 't Hooft model?
- ☆ Generalisations: Other coset models (Kiritsis) and supersymmetric versions.
- ☆ Embedding in String theories: A solvable sub-sector within theories such as D1-D5?

The end